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Modeling of Decision-Making Based on Hesitant Fuzzy Sets

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Abstract

The hesitant fuzzy set (HFS) is successfully used as a fuzzy set (FS) extension to demonstrate circumstances where it is permissible to determine a few potential membership de- grees (MDs) of a component in a set due to the uncertainty between distinct values. Aggregation operators (AOs) are widely applied to accumulate vague and uncertain information these days. The hesitant fuzzy weighted aggregation geometric (HFWAG) operator is created by the authors as a new AO for hesitant fuzzy (HF) data. Some of the most important qualities of the suggested operators are highlighted, as well as their extensive interrelationship. Then, using those operators, we develop a methodology for interpreting the HF multiple criteria decision making (MCDM) hurdles. The proposed concepts are demonstrated using a real-life example related to medical di- agnosis, with the results' believability tested using test criteria and compared. Because the weights of criteria can affect the outcome of a choice, they have a substantial impact on judgments. The De Luca-Termini entropy metric is utilized to estimate the weights of criteria in the current work. Moreover, TOPSIS and VIKOR were also employed to compare the results of our method.

Keywords: Fuzzy sets, hesitant fuzzy sets, aggregation operator.

1. Introduction

Hesitation is a common issue involves in the decision-making process in our daily life. Torra (Torra, 2010) presented the idea of HFSs, the further extension of fuzzy sets (FSs) (Zadeh, 1965) to tackle such problems. Many real-life issues include hesitancy. Torra (Torra, 2010) investigated the HFS as a strong version of the FS that permits several positive grades to be connected with each preference information for coping with a difficult and confusing circumstance in order to tackle such challenges. Researchers widely studied the notion for pattern recognition (Qian, 2013), medical diagnosis (Szmidt, 2020), similarity measures (Tian, 2019), and decision making (Ullah, 2019) difficulties, keeping the benefits of hesitant fuzzy sets in mind. Nowadays, HFSs are attaining more attraction from the researches because of its specification. HFSs consist of number of distinct real values between 0 and 1 as a member- ship degree. Consequently, HFSs can handle vague and uncertain information more efficiently and conveniently than other extensions of FSs. Moreover, decision makers (DMs) have pro- posed various extensions of HFSs for example, dual HFSs (DHFSs) (Zhu, 2012), generalized HFSs (Qian, 2013), trapezoidal-valued HFSs (T. Rashid, 2014), multi-HFSs (Peng, 2015) and intuitionistic-valued HFSs (Wang, 2016). Later on, to handle the hesitant fuzzy (HF) information Xia and Xu (Xia, 2011) introduced the concept of HF elements (HFEs) and then presented some basic arithmetic operations over HFEs.

The concept of entropy was first presented by Clausius (Clausius, 1865) in 1865 that measures the level of uncertainty or vagueness present in the data. The recent past has shown the application of entropy in various fields of life for instance, accounting, statistics, finance, economics, pattern recognition, computer sciences, fuzzy sets, etc. Later on, Shanon proposed the concept of Shanon entropy in 1948 through which the DMs can measure the uncertainty of occurrence of certain event. Evalu- ating criteria weights based on entropy measure is an extensively used technique. Entropy is an imperative instrument for determining vague and uncertain information. The larger entropy value indicates larger vagueness in the data. The HFS is primarily utilized to tackle with vague, un- predictable data. Therefore, use of entropy to evaluate weights is suitable under the frame work of HFSs. MCDM problems are a big element of human culture, and they're used a lot in real- life situations like economics, management, and engineering. The goal of the MCDM method is to discover the best option from the available options. With the advancement of science and technology, uncertainty has become an increasingly important aspect in decision-making (DM) analyses. Furthermore, in order to acquire correct data, the involvement of decision-makers along the process is quite difficult. The majority of the data gathered from numerous sources is either ambiguous or imprecise, resulting in erroneous outcomes. Using this imprecise or ambiguous information, the DM process' goal is to select the best items from the available. To this end, a number of notions have been used from time to time to arrive at the proper conclusions by leveraging characteristics such as crispness, determinism, and precision in nature. However, a notion of FSs and its expansions have been developed to deal with data ambiguity. MCDM is a discipline that assists DMs in making an optimal decision from alternatives based on multiple criteria (Antucheviciene, 2015; Wiecek, 2008). Various MCDM strategies have been developed in the current decay (Mardani, 2015; Tzeng, 2011) and have been used in various fields, such as supplier selection (Kannan, 2014; Liu, 2011) and development of man- aging energy projects (Cristobal, 2011). To address the MCDM issues, Sindhu et al. presented an assessment model (Sindhu, 2021; Sindhu, 2019) based on TOPSIS for PFSs.

Aggregation operators (AOs) are the tool used in decision-making process that fuses the infor- mation into a single esteem. There are number of Aos (Akram, 2020; Ali, 2021; Kamaci, 2021; Sindhu, 2021; Wang, 2020; Zulqarnain, 2021; Zhou, 2019; Zhou, 2019) available in the literature developed by different researchers.

In the present work, a novel AO named HF weighted averaging geometric (HFWAG) operator is developed to aggregate the HF information for the decision-making process. The proposed HFWAG operator has the following characteristics:

HFWAG is simple and easy to handle the HFEs,

HFWAG consider the interrelationship among the HF data.

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VIKOR method was discovered by Opricovic in 1998 (Opricovic, 1998) for the control and optimization of the multi-criteria decision-making process. Nowadays, VIKOR (Kumar, 2021; Lam, 2021; Wang , 2021; Wu, 2021) has been broadly using in the decision-making process to resolve the MCDM problems. It is a type of relieving method where the rankings can fluctuate of a finite decision by benefiting the group and minimizing the regret. The main idea which VIKOR encapsulates is to ascertain the productive outcome or PIS and NIS and then choosing the optional solution in accordance with the degree of position and the evaluation value. Thus designated and PIS with limitations to benefits acceptable and equaling the process. Solutions that are gained as a result of the VIKOR technique are a com- promise and near to the optimal solution. The VIKOR algorithm getting a compromise solution that is usually favored through decision-makers by learning in the side of the advantages and distancing from personal losses.

The fuzzy TOPSIS method (Jasiulewicz-Kaczmarek, 2021; Sindhu, 2021; Sindhu, 2019; Wang, 2021) is also utilizing to obtain the final grading. Hwang and Yoon (Hwang, 1981) first proposed the TOPSIS through which an alternative is picked so that it has the shortest distance from the PIS and the farthest distance from the NIS. TOPSIS is essential to know the MCDM technique to deal with problems of actual life. This idea has mainly been studied here under the framework of HFSs.

The main objective of the present article is to choose the best surgical mask in the current pan- demic situations by using the proposed AO based on HF environment in Pakistan. An MCDM model based on HFWAG operator and newly established score function is proposed to resolve the multiple criteria problem and then a comparative analysis with existing techniques like VIKOR and TOPSIS. From the outcomes obtained, we see that the proposed MCDM model is user friendly, easy to evaluate and has less time complexity than VIKOR and TOPSIS techniques.

Rest of the article is planed as follows. Section 2 probes some fundamentals about HFSs. An novel AO and score function to compare HFEs are proposed in Section 3. A comprehensive process to evaluate the weights of criteria is presented in Section 4. Based on proposed AO and score function an MCDM model is developed in Section 5 to reach the final decision. In Section 6, an practical example related to the current pandemic situation resolved by using the proposed MCDM model under HF framework. Moreover, the same data is resolved by using existing approaches like fuzzy VIKOR and fuzzy TOPSIS in Section 7. In Section 8, a comprehensive comparative analysis with existing techniques is performed to validate and effectiveness of the proposed model. Conclusions are drawn in Section 9.

2. Basic Knowledge

In this section, we defined FSS, IFSs, concepts of HFSs, HFE, Shanon entropy, fuzzy entropy, and operations on HFSs.

Definition 2.1.. Suppose $X = \{x_1, x_2, x_3, ..., x_n\}$ be the universal set. Fuzzy set in X is defined by $\mu_B(\tilde{x}) : X \rightarrow [0, 1]$ is the membership function of B and it can be written as given below:

$$B = \{(\tilde{x}, \mu_B(\tilde{x})), \tilde{x} \in X\},$$

where $\mu_B(\tilde{x})$ is the degree of membership of $\tilde{x}$ in B and each pair $(\tilde{x}, \mu_B(\tilde{x}))$ is singleton.

Example 2.1.. Let $X = \{\tilde{l}, \tilde{m}, \tilde{n}\}$ and $\mu_B(\tilde{x}) : X \rightarrow [0, 1]$ defined by $\mu_B(\tilde{l}) = 0.2$, $\mu_B(\tilde{m}) = 0.4$, $\mu_B(\tilde{n}) = 0.5$ then fuzzy set can be written as

$$B = \{(\tilde{l}, 0.2), (\tilde{m}, 0.4), (\tilde{n}, 0.5)\}.$$

Definition 2.2.. Suppose X be a reference set, then the intuitionistic fuzzy sets (IFSs), E on X is represented as (Z. Ali, 2021; A. Alinezhad, 2011; V. Torra, 2010).

$$E = \{< x_i, \phi_A(x_i), \varphi_A(x_i) > \text{ for } x_i \in X\},$$

Besides, the membership degree and the non-membership degree are shown by $\phi_A(x_i)$ and by $\varphi_A(x_i)$, respectively. Accordingly, the following relation should be satisfied:

$$0 \leq \phi_A(x_i) + \varphi_A(x_i) \leq 1 \text{ for } x_i \in X.$$

Definition 2.3.. Suppose X be the universal set, then an HFSs as B on X is defined by function $\rho_B(\tilde{x})$ that X returns to a subset of [0,1]. Xia and Xu (M. M. Xia, 2011) indicated the HFSs mathematically as given below:

$$B = \{< \tilde{x}, \rho_B(\tilde{x}) > | \tilde{x} \in X\},$$

where $\rho_B(\tilde{x})$ is defining as a set of membership degree for an element under a subset of [0,1], indicating the membership degree of an element $\tilde{x} \in X$. For convenience, HFE can be expressed as $h_f(x) = \rho_B(\tilde{x}$

Definition 2.4. (M. M. Xia, 2011). Let $h = h_f(x) = \rho_B(\tilde{x}$, where $h_f(x)$ a non-empty and finite subset of [0, 1] is a collection of elements called hesitant fuzzy element (HFE).

Definition 2.5. Let $h_f(x)$, $h^1_f(x)$ and $h^2_f(x)$ be three HFEs then some basic operations are defined as:

(1) Union: $h^1_f \cup h^2_f(x) = \{\gamma \in h^1_f \cup h^2_f(x) | \gamma \geq max\{\{h^1_f(x)\}, \{h^2_f(x)\}\}\}$;

(2) Intersection: $h^1_f \cap h^2_f(x) = \{\gamma \in h^1_f \cup h^2_f(x) | \gamma \leq min\{\{h^1_f(x)\}, \{h^2_f(x)\}\}\}$;

(3) Complement: $h^c_f(x) = \{\{1 - \gamma | \gamma \in h_f\}\}$;

(4) Exponent: $h^\lambda_f(x) = \{\{\gamma^\lambda | \gamma \in h_f(x)\}\}$;

(5) Scalar multiple: $\lambda h_f(x) = \{\{1 - (1 - \gamma)^\lambda | \gamma \in h_f(x)\}\}$;

(6) Sum: $(h_f^1 \oplus h_f^2)(x) = \{\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 | \gamma_1 \in h_f^1(x), \gamma_2 \in h_f^2(x)\}\}$;

(7) Product: $(h_f^1 \otimes h_f^2)(x) = \{\{\gamma_1\gamma_2 | \gamma_1 \in h_f^1(x), \gamma_2 \in h_f^2(x)\}\}$.

Example 2.2. Let $X = \{x_1, x_2, x_3\}$ be a fixed set, HFSs h_f^1 and h_f^2 on X be

$h_f^1 = \{< x_1, \{0.2, 0.3\} >, < x_2, \{0.5, 0.7\} >, < x_3, \{0.2, 0.3, 0.4, 0.6\} >\}$ and $h_f^2 = \{< x_1, \{0.4, 0.5\} >, < x_2, \{0.3, 0.4\} >, < x_3, \{0.1, 0.2, 0.3, 0.5\} >\}$, respectively. Then we have

$$(1)\ (h_f^1 \cup h_f^2)(x_3) = \{\gamma \in h_f^1(x_3) \cup h_2(x_3) | \gamma \geq max\{h_f^1(x_3), (h_f^2(x_3)\}$$
$$= \{\gamma \in h_f^1(x_3) \cup h_f^2(x_3) | \gamma \geq max\{0.2, 0.1\}\} = \{0.2, 0.3, 0.4, 0.5, 0.6\};$$
$$(2)\ (h_1 \cap h_f^2)(x_3) = \{\gamma \in h_1(x_3) \cup h_2(x_3) | \gamma \leq min\{(h_f^1(x_3), (h_f^2(x_3)\}$$
$$= \{\gamma \in h_f^1(x_3) \cup h_f^2(x_3) | \gamma \leq min\{0.6, 0.5\}\} = \{0.1, 0.2, 0.3, 0.4, 0.5\};$$

(5) $(h_f^1 \otimes h_f^2)(x_1) = \{\gamma_1\gamma_2 | \gamma_1 \in h_f^1(x_1), \gamma_2 \in h_f^2(x_1)\} = \{0.08, 0.1, 0.12, 0.15\}$.

Definition 2.6. (De Luca, 1972). Let $A_n = \{P = (p_1, p_2, ..., p_n) : 0 \geq p_i \leq 1, \sum_{i=1}^{n} P_i = 1\}$ be an n-complete set of probability distributions irrespective of the probability distribution $P = (p_1, p_2, ..., p_n) \in A_n$ Shanon entropy is defined as follows:

$$H(P) = -\sum_{i=1}^{n} p(x_i) \log p(x_i).$$

Definition 2.7. If A is an HFS specified in the world of discourse X, then A's fuzzy entropy is as follows:

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n} [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i) \log(1 - \mu_A(x_i))].$$

3. Propose aggregation operator and score function

In the present section a HFWAG operator based on HFWA, HF weighted geometric (HFWG) operators is proposed. Some theorems related to the proposed HFWAG operator are also penned. Further, functions for score and accuracy are established to comparison data.

Definition 3.1. Let $h_f^j (j = 1, 2, 3, ..., n)$ symbolize the set of HFEs. A mapping operator is an HFWG operator $H^n \rightarrow H$

$$\text{HFWG}(h_f^1, h_f^2, ..., h_f^n) = \cup_{\gamma_1 \in h_f^1, \gamma_2 \in h_f^2, \gamma_3 \in h_f^3, ..., \gamma_n \in h_f^n} \{\textstyle\prod_{j=1}^{n} \gamma_j^{w_j}\}.$$

Definition 3.2.. Let $h_f^j (j = 1, 2, 3, ..., n)$ be the collection of HFEs. A HFWA operator is a mapping $H^n \rightarrow H$

$$\text{HFWA}(h_f^1, h_f^2, ..., h_f^n) = \cup_{\gamma_1 \in h_f^1, \gamma_2 \in h_f^2, \gamma_3 \in h_f^3, ..., \gamma_n \in h_f^n} \{1 - \textstyle\prod_{j=1}^{n} (1 - \gamma_j)^{w_j}\}.$$

Definition 3.3. Let $h_j (j = 1, 2, 3, ..., n)$ be the collection of HFEs. A hesitant fuzzy weighted averaging geometric (HFWAG) operator is a mapping $H^n \rightarrow H$

$$\text{HFWAG}(h_f^1, h_f^2, ..., h_f^n) = \cup_{\gamma_1 \in h_f^1, \gamma_2 \in h_f^2, \gamma_3 \in h_f^3, ..., \gamma_n \in h_f^n} \{1 - \textstyle\prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j}\}.$$

Lemma 3.1. *Let* $x_j > 0,\ w_j > 0,$ *j=1, 2, 3, ...,n and* $\sum_{j=1}^{n} w_j = 1$, *then* $\prod_{j=1}^{n} x_j^{w_j} \leq \sum_{j=1}^{n} w_j x_j$ *with equality iff* $x_1 = x_2 = x_3 = \ldots = x_n$.

Theorem 3.1. *Let* $h_f^j(j = 1, 2, 3, \ldots, n)$ *be the collection of HFEs having the weight vectors* $w = (w_1, w_2, w_3, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$ *then*
HFWG$(h_f^1, h_f^2, \ldots, h_f^n) \leq$ *HFWA*$(h_f^1, h_f^2, \ldots, h_f^n)$
Proof. *For any* $\gamma_1 \in h_f^1, \gamma_2 \in h_f^2, \gamma_3 \in h_f^3, \ldots, \gamma_n \in h_f^n$ *based on lemma 3.1, we have*
$\prod_{j=1}^{n} \gamma_j^{w_j} \leq \sum_{j=1}^{n} w_j \gamma_j$
Replacing γ_j *by* $1 - \gamma_j$ *from the above equation*
$\prod_{j=1}^{n} (1 - \gamma_j)^{w_j} \leq \sum_{j=1}^{n} w_j (1 - \gamma_j)$
we take minus on both sides
$-\prod_{j=1}^{n} (1 - \gamma_j)^{w_j} \geq -\sum_{j=1}^{n} w_j (1 - \gamma_j)$
Adding 1 on both sides
$1 - \prod_{j=1}^{n} (1 - \gamma_j)^{w_j} \geq 1 - \sum_{j=1}^{n} w_j (1 - \gamma_j)$
$\implies 1 - \sum_{j=1}^{n} w_j (1 - \gamma_j) \leq 1 - \prod_{j=1}^{n} (1 - \gamma_j)^{w_j}$
$\implies$ *HFWG*$(h_1, h_2, h_3, \ldots, h_n) \leq$ *HFWA*$(h_1, h_2, h_3, \ldots, h_n)$.

Theorem 3.2. *Let* $h_f^j(j = 1, 2, 3, \ldots, n)$ *be the collection of HFEs having the weight vectors* $w = (w_1, w_2, w_3, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$ *then prove that*
$1 - \prod_{j=1}^{n} \gamma_j^{w_j} (1 - \gamma_j)^{w_j} \leq 1 - \prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j}$.
Proof. *For any* $\gamma_1 \in h_f^1, \gamma_2 \in h_f^2, \gamma_3 \in h_f^3, \ldots, \gamma_n \in h_f^n$, *based on lemma 3.1, we have*
$\prod_{j=1}^{n} \gamma_j^{w_j} \leq \sum_{j=1}^{n} w_j \gamma_j$
Replacing γ_j *by* $\gamma_j(1 - \gamma_j)$ *from the above equation*
$\prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j} \leq \sum_{j=1}^{n} w_j (\gamma_j (1 - \gamma_j))$
we take minus on both sides
$-\prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j} \geq -\sum_{j=1}^{n} w_j (\gamma_j (1 - \gamma_j)$
Adding 1 on both sides
$1 - \prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j} \geq 1 - \sum_{j=1}^{n} w_j (\gamma_j (1 - \gamma_j))$
As we know that $-\sum_{j=1}^{n} w_j x_j \geq -\prod_{j=1}^{n} x_j^{w_j}$
$1 - \prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j} \geq 1 - \sum_{j=1}^{n} w_j (\gamma_j (1 - \gamma_j))$
$1 - \prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j} \geq 1 - \sum_{j=1}^{n} w_j \gamma_j (1 - \gamma_j)$
$\implies 1 - \prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j} \geq 1 - \prod_{j=1}^{n} \gamma_j^{w_j} (1 - \gamma_j)$
$\implies 1 - \prod_{j=1}^{n} \gamma_j^{w_j} (1 - \gamma_j) \leq 1 - \prod_{j=1}^{n} (\gamma_j (1 - \gamma_j))^{w_j}$. *Hence proved*

Theorem 3.3. *Let* $h_f^j(j = 1, 2, 3, \ldots, n)$ *be the collection of HFEs having the weight vectors* $w = (w_1, w_2, w_3, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$ *then,*
HFWG$(h_f^1, h_f^2, \ldots, h_f^n) \leq$ *HFWA*$(h_f^1, h_f^2, \ldots, h_f^n) \leq$ *HFWAG*$(h_f^1, h_f^2, \ldots, h_f^n)$.
Proof. *For any* $\gamma_1 \in h_f^1, \gamma_2 \in h_f^2, \gamma_3 \in h_f^3, \ldots, \gamma_n \in h_f^n$, *As we have already proved Theorem 3.1*
HFWG$(h_f^1, h_f^2, \ldots, h_f^n) \leq$ *HFWA*$(h_f^1, h_f^2, \ldots, h_f^n)$ *clearly we say that*
HFWG$(h_f^1, h_f^2, \ldots, h_f^n) \leq$ *HFWA*$(h_f^1, h_f^2, \ldots, h_f^n) \leq$ *HFWAG*$(h_f^1, h_f^2, \ldots, h_f^n)$.

Definition 3.4. To compare the HFEs a score ($S_f^i\, h(x)$) and accuracy functions are defined as follows:

$$S_f^i h_f(x)) = 1 - \frac{\sum_{j=1}^{n} h_{ij}}{n}, \quad S_f^i h_f(x)) \in [-1, 1]$$

and the accuracy function is

$$A_f^i h_f(x)) = 1 + \frac{\sum_{j=1}^{n} h_{ij}}{n}, \quad A_f^i h_f(x)) \geq 1.$$

Definition 3.5. On the basis of functions for score and accuracy of HFEs, the comparison be-
Two h_f^1 and h_f^2 is defined as follows:

- If $S_f^i(h_f^1) > S_f^i(h_f^2)$, then $h_f^1 \succ h_f^2$, where $\succ$ is refer as "superior to";
  If $S_f^i(h_f^1) = S_f^i(h_f^2)$,
- $(a) A_f^i(h_f^1) > A_f^i(h_f^2)$, then $h_f^1 \succ h_f^2$,
  $(b) A_f^i(h_f^1) = A_f^i(h_f^2)$, then $h_f^1 \sim h_f^2$, $\sim$ is refer as "equivalent to".

4. Determination of the weights of the criteria based on Shanon entropy

The ambiguities surrounding decision-making challenges highlight the fact that the informa- tion acquired concerning weight-related factors is frequently anonymous or incomplete. As a result of not knowing the weights of the criterion, this study relies on the objective weight tech- nique to evaluate the criteria's weights. De Luca and Termini cite5 offer an entropy technique that provides a perspective within the framework of HFSs. Because it includes a classical entropy technique for the circumstances of expert views, this study used the entropy weights approach based on the De Luca and Termini entropy. (A. De Luca, 1972) provided a non-probabilistic entropy formula of FSs based on Shanon function on a finite universal sets X = $\{x_1, x_2, x_3, ..., x_n\}$.

$$E_{LT} = -K\sum_{i=1}^{n} [\rho_B(x_i) ln \rho_B(x_i) + (1-\rho_B(x_i)) ln(1-\rho_B(x_i))], K > 0,$$

where $\rho_B : X \to [0,1]$ and K is a positive constant.

The following are the steps to evaluate the weights of criteria weights based on De Luca-Termini entropy method:

Step 1. Form a HF decision matrix (DM_r)$S = (s_{ij})_{m\times n}$.

Step 2. Determine the normalized HF DM_r $S' = (s'_{ij})_{m\times n}$, where

$$s'_{ij} = \frac{\sum_{i=1}^{m} s_{ij}}{|s_{ij}|} \tag{1}$$

Step 3. Evaluate the De Luca-Termini normalized entropy E_j with the help of formula below.

$$E_j = -\frac{1}{m ln 2}\sum_{i=1}^{m} (s'_{ij} ln(s'_{ij}) + (1-s'_{ij})(1-ln(s'_{ij}))), j = 1,2,3,..,m. \tag{2}$$

Step 4. Compute the weights of criteria W_j as:

$$W_j = \frac{1-Ej}{\sum_{j=1}^{n}(1-Ej)}, j = 1,2,3,...,m. \tag{3}$$

5. MCDM model based on Proposed Aggregation Operator

The following procedures are necessary to arrive at the optimal solution, according to pro- posed AO HFWAG.

Step 1. Determine criteria's weights by using the technique presented in subsection 4.

Step 2. With the help of proposed HFWAG operator an accumulated decision matrix $B = (\daleth_{ij})_{m\times n}$ is constructed so that

$$B = \begin{bmatrix} \daleth_{11} & \daleth_{12} & \daleth_{13} & \daleth_{14} \\ \daleth_{21} & \daleth_{22} & \daleth_{23} & \daleth_{24} \\ \daleth_{31} & \daleth_{32} & \daleth_{33} & \daleth_{34} \\ \daleth_{41} & \daleth_{42} & \daleth_{43} & \daleth_{44} \end{bmatrix},$$

where, $\daleth_{ij} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \gamma_3 \in h_3, \ldots, \gamma_n \in h_n} \{ \prod_{j=1}^{n} (\gamma_{ij}(1 - \gamma_{ij}))^{w_j} \}$.

Step 3. Based on Definition3.4, evaluate $S^i_f h(x)$.

Step 4. On the basis of values obtained in Step 3, rank the alternatives from largest to lowest values of $S^i_f h(x)$.

Step 5. By following the Step 4, pickup the best alternative.

5.1. Practical Example

In the current pandemic (Covid-19) situation different types of mask are being used by the people to prevent from this fetal virus. According to the world health organization (WHO) if a person wears the mask, he/she is secured from Covid-19 by 95 percent. There are number of surgical masks like, N 95 (M_1), N 99 (M_2), P 99 (M_3) and cloth mask with filter (M_4) available in the market. These masks are investigated under the four criteria such as, filter efficiency (C_1), airflow rate (C_2), environmental affect (C_3) and face seal fit (C_4). Evaluation of efficiency of available surgical masks are carried by fuzzy TOPSIS, fuzzy VIKOR and proposed approaches. All the HF information collected shown in Table 1.

1.1. **Determination of Weights of Criteria.** In order to evaluate the weights of criteria the following steps are adopted.

Step 1. Form a HF decision matrix $S = (s_{ij})_{4\times4}$.

Step 2. Based on Eq. 1, Table 2 shows the normalized HF decision matrix $S' = (s'_{ij})_{m4\times4}$.

Table 1: HF decision matrix

Alternatives	C_1	C_2	C_3	C_4
M_1	{0.4, 0.3, 0.1}	{0.9, 0.8, 0.7, 0.1}	{0.9, 0.6, 0.5, 0.3}	{0.5, 0.4, 0.3}
M_2	{0.5,0.4}	{0.9,0.7,0.6,0.3}	{0.7,0.4,0.3}	{0.6,0.5}
M_3	{0.3,0.2,0.1}	{0.9,0.6}	{0.8,0.7,0.7}	{0.7,0.4,0.1}
M_4	{0.2,0.1}	{0.8,0.7,0.5,0.3}	{0.9,0.8,0.6}	{0.8,0.5,0.4}

Table 2: Normalized HF decision matrix

Alternatives	C_1	C_2	C_3	C_4
M_1	0.26	0.625	0.575	0.4
M_1	0.45	0.625	0.466	0.55
M_1	0.2	0.75	0.75	0.4
M_1	0.15	0.575	0.766	0.566

Step 3. With the help of Eq. 2, evaluate the De Luca-Termini normalized entropy E_j as follows:

$E_1 = 0.7682, E_2 = 0.80901, E_3 = 0.80183, E_4 = 0.80462.$

$1 - E_1 = 1 - 0.7682 = 0.2318,$

$1 - E_2 = 1 - 0.80901 = 0.19098,$

$1 - E_3 = 1 - 0.80183 = 0.19816,$

$1 - E_4 = 1 - 0.80462 = 0.19537$.

Step 4. Based on Eq. 3, the weights of criteria W_j are:

$W_1 = 0.2833, W_2 = 0.2341, W_3 = 0.2429, W_4 = 0.2395$, so that,

1.1.1. $\sum_{i=1}^{4} \mathrm{Wi} = 1$. *Solution of Practical Example Based on*

1.1.2. *Proposed MCDM Model.* The proposed MCDM model has the following steps.

Step 1. Weights of criteria by using the technique described in Section 4 are:

$W_1 = 0.2833, W_2 = 0.2341, W_3 = 0.2429, W_4 = 0.2395$, so that, $\sum_{i=1}^{4} (W_i) = 1$.

Step 2. With the help of proposed HFWAG operator an accumulated decision matrix $B = (\kappa_{ij})_{m\times n}$ is constructed so that

Based on HFWAG operator the amassed hesitant fuzzy decision matrix is attained as:

$\kappa_{11} = 1-(0.4(1-0.4))^{0.2833}\times(0.3(1-0.3))^{0.2342}\times(0.1(1-0.1))^{0.2429}\times(0.1(1-0.1))^{0.2395} = 0.8550$,

$\kappa_{12} = 1-(0.9(1-0.9))^{0.2833}\times(0.8(1-0.8))^{0.2342}\times(0.7(1-0.7))^{0.2429}\times(0.1(1-0.1))^{0.2395} = 0.8734$,

$\kappa_{13} = 1-(0.9(1-0.9))^{0.2833}\times(0.6(1-0.6))^{0.2342}\times(0.5(1-0.5))^{0.2429}\times(0.3(1-0.3))^{0.2395} = 0.8221$,

$\kappa_{14} = 1-(0.5(1-0.5))^{0.2833}\times(0.4(1-0.4))^{0.2342}\times(0.3(1-0.3))^{0.2429}\times(0.3(1-0.3))^{0.2395} = 0.7723$.

Similarly other entries can be computed and the accumulated matrix B is obtained as:

$$B = \begin{bmatrix} 0.8550 & 0.8734 & 0.8221 & 0.7723 \\ 0.7571 & 0.8293 & 0.7833 & 0.7528 \\ 0.86906 & 0.8182 & 0.8055 & 0.8560 \\ 0.8940 & 0.7971 & 0.8346 & 0.7839 \end{bmatrix}$$

Step 3. Based on Definition 3.4, the values of $S^i_f h(x)$ are:

$$S_f^1 = 0.1693,\ S_f^2 = 0.219375,\ S_f^3 = 0.162825,\ S_f^4 = 0.1726.$$

Step 4. Ranking order of the alternatives based on $S^i_f h(x)$ is, $M_2 \succ M_4 \succ M_3 \succ M_1$.

Step 5. The best alternative obtained is M_2.

5.2. Comparison with Existing Techniques

For the comparison purpose fuzzy VIKOR and fuzzy TOPSIS are utilized to resolve the MCDM problem.

1.2. **Fuzzy VIKOR Technique.** Fuzzy VIKOR technique is utilized on the following way:

Step1. The weights of criteria Wj are:

$W_1 = 0.2833, W_2 = 0.2341, W_3 = 0.2429, W_4 = 0.2395$, so that, $\sum_{i=1}^{4} (W_i) = 1$.

Step 2. Based on HFWAG operator the amassed hesitant fuzzy decision matrix is attained as:

$$B = \begin{bmatrix} 0.8550 & 0.8734 & 0.8221 & 0.7723 \\ 0.7571 & 0.8293 & 0.7833 & 0.7528 \\ 0.86906 & 0.8182 & 0.8055 & 0.8560 \\ 0.8940 & 0.7971 & 0.8346 & 0.7839 \end{bmatrix}$$

Step 3.

(1) The PIS κ_j^+ and NIS κ_j^- are obtained $\kappa^+ = [\kappa_1^+, \kappa_2^+, \kappa_3^+, \kappa_4^+] = [0.8940, 0.8734, 0.8346, 0.8560]$.

$\kappa^- = [\kappa_1^-, \kappa_2^-, \kappa_3^-, \kappa_4^-] = [0.7571, 0.7971, 0.7833, 0.7528]$.

(2) The values of group utility S_i are: $S_1 = 0.3340, S_2 = 0.901, S_3 = 0.3586, S_4 = 0.4014$.

(3) The values of indivisible regret are:
$R_1 = 0.1942, R_2 = 0.2833, R_3 = 0.1693, R_4 = 0.2341$.

(4) The values of VIKOR index $Q_i (i = 1, 2, 3, 4)$ for different values of μ are presented in Table 3. From Table 3, we see that the best alternative obtained is M_2 that is N 99 be the best surgical mask.

Table 3: Values of Q_i for different values of μ

Alternatives	$Q_i(\mu = 0.1)$	$Q_i(\mu = 0.5)$	$Q_i(\mu = 0.55)$	$Q_i(\mu = 0.7)$	$Q_i(\mu = 0.9)$
M_1	1	1	1	1	1
M_2	-0.1	-0.5	-0.55	-0.7	-0.9
M_3	1.2471	1.1180	1.1018	0.5683	0.9889
M_4	0.5850	0.7166	0.9346	0.7167	0.8482
Ranking order	$M_2 \succ M_4 \succ M_1 \succ M_3$	$M_2 \succ M_4 \succ M_1 \succ M_3$	$M_2 \succ M_4 \succ M_1 \succ M_3$	$M_2 \succ M_3 \succ M_4 \succ M_1$	$M_2 \succ M_4 \succ M_3 \succ M_1$

(5) Now analyze the conditions to check the ranking order of the alternatives: For $\mu = 0.1$
C_1: $|Q(A^2) - Q(A^1)| = |-0.1 - 0.5850| = 0.6850 \geq \frac{1}{3}$,
which shows that C_1 is satisfied and hence, M_2 is the best surgical mask. For $\mu = 0.5$
C_1: $|Q(A^2) - Q(A^1)| = |-0.5 - 0.7166| = 1.2166 \geq \frac{1}{3}$,
means that C_1 is satisfied and hence, M_2 is the best surgical mask. For $\mu = 0.55$
C_1: $|Q(A^2) - Q(A^1)| = |-0.55 - 0.9346| = 1.4846 \geq \frac{1}{3}$,
that is C_1 is satisfied and hence, M_2 is the best surgical mask. For $\mu = 0.7$
C_1: $|Q(A^2) - Q(A^1)| = |-0.7 - 0.7166| = 1.2683 \geq \frac{1}{3}$,
shows that C_1 is satisfied and hence, M_2 is the best surgical mask. For $\mu = 0.9$
C_1: $|Q(A^2) - Q(A^1)| = |-0.9 - 0.8482| = 1.7486 \geq \frac{1}{3}$,
that is C_1 is satisfied and hence, M_2 is the best surgical mask.
It is observed that the sensitivity analysis (SA) based on μ reveals that the favorable mask is $N99$.

Figure 1: Ranking of alternatives by VIKOR

Fig. 1, a radar diagram indicates the SA of the obtained outcomes which shows the stability of the ranking order of the alternatives. According to the SA, it illustrates that M_2 has the maximum significance in the set of alternatives.

1.3. **Fuzzy TOPSIS Method.** Fuzzy TOPSIS has the following procedure:

(1) Based on PIS κ_j^+ and NIS κ_j^-, the values of d_i^+ and d_i^- of each alternatives $M_i(i = 1, 2, 3, 4.)$ are:
$d_1^+ = 0.1262, d_2^+ = 0.1824, d_3^+ = 0.1604, d_4^+ = 0.1500.$
Similarly,
$d_1^- = 0.12622, d_2^- = 0.12506, d_3^- = 0.1399, d_4^- = 0.1503.$

(2) The values of relative closeness coefficient δ_i of every alternative M_i to the hesitant fuzzy PIS κ^+ is presented in Table 4.

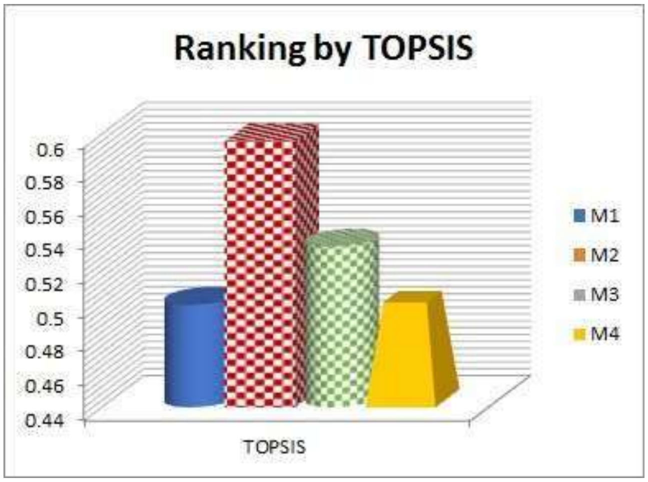
(3) We get, $\delta_2 >_j \delta_3 > \delta_1 > \delta_4$, so the ranking order become $M_2 \succ M_3 \succ M_1 \succ M_4$.

Hence, M_2 is the best alternative obtained from Fuzzy TOPSIS which is also represented graph- ically in Fig. 2.

Table 4: Ranking Order by TOPSIS

Alternatives	values of $\delta_i (i = 1, 2, 3, 4)$
M_1	0.5
M_2	0.5957
M_3	0.5341
M_4	0.4995

Figure 2: Ranking of alternatives by TOPSIS



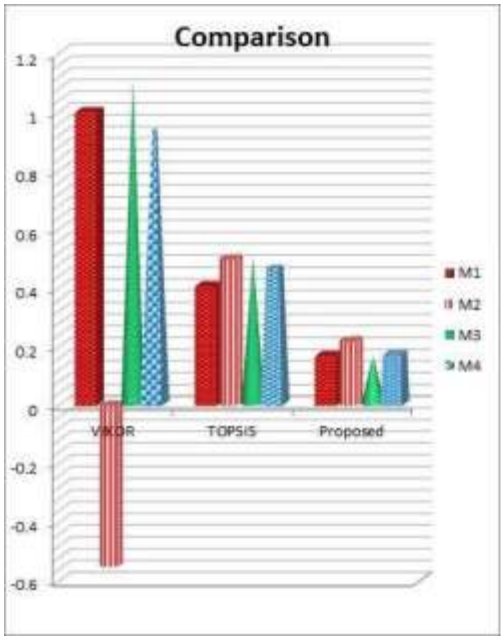
6. Comparative Analysis

The authors have studied a realistic case relating to the current pandemic (Covid-19) setting in this study. To select the best mask from the set of masks, the suggested MCDM is used. The findings of the new MCDM model were then compared to those produced using current approaches such as fuzzy TOPSIS and fuzzy VIKOR. The findings in Table 5 show that the best alternative provided by the proposed MCDM model corresponded with the fuzzy TOPSIS and fuzzy VIKOR, indicating that the novel MCDM model has been validated. Furthermore, a SA was done by giving several values to *mu* in Table 3, and the best option was the same. The suggested, fuzzy VIKOR, and fuzzy TOPSIS techniques are depicted graphically in Fig. 3.

Table 5: Comparison of Ranking Order

Alternatives	VIKOR Technique	TOPSIS	Proposed Technique
M_1	1	0.4067	0.1693
M_2	-0.55	0.499	0.2194
M_3	1.1019	0.5	0.1628
M_4	0.9346	0.4653	0.1726

Figure 3: Ranking of alternatives



7. Conclusion

In the present work, we introduced the AO named HFWAG to tackle the HF information. Also, score and accuracy functions are presented to compare the HFEs. Weights of criteria have great impact in decision-making process, Shanon entropy is used to determine the weights of crite- ria to avoid the biasedness. Subsequently, a HF MCDM model based on the proposed HFWAG operator has been provided. At the end, a same practical example related the current pandemic issue is examined with the help of proposed MCDM model, fuzzy VIKOR, fuzzy TOPSIS and investigated that the outcomes obtained are coincide with each other. The comparative analy- sis represents the validity and effectiveness of the proposed MCDM model. The comparative analysis with other techniques shows the advantages, validity and effectiveness of the proposed MCDM model. For future work, the proposed HFWAG operator will be extended to resolve the MCDM issues based on bipolar interval-valued hesitant information and bipolar Pythagorean fuzzy environment.

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